Higher-order Occurrence Pooling for Bags-of-Words: Visual Concept Detection

Piotr Koniusz, Fei Yan, Philippe-Henri Gosselin, Krystian Mikolajczyk

Abstract—In object recognition, the Bag-of-Words model assumes: i) extraction of local descriptors from images, ii) embedding the descriptors by a coder to a given visual vocabulary space which results in mid-level features, iii) extracting statistics from mid-level features with a pooling operator that aggregates occurrences of visual words in images into signatures, which we refer to as First-order Occurrence Pooling. This paper investigates higher-order pooling that aggregates over co-occurrences of visual words. We derive Bag-of-Words with Higher-order Occurrence Pooling based on linearisation of Minor Polynomial Kernel, and extend this model to work with various pooling operators. This approach is then effectively used for fusion of various descriptor types. Moreover, we introduce Higher-order Occurrence Pooling performed directly on local image descriptors as well as a novel pooling operator that reduces the correlation in the image signatures. Finally, First-, Second-, and Third-order Occurrence Pooling are evaluated given various coders and pooling operators on several widely used benchmarks. The proposed methods are compared to other approaches such as Fisher Vector Encoding in the same testbed and achieve state-of-the-art results.

Index Terms—Bag-of-Words, Mid-level features, First-order, Second-order, Co-occurrence, Pooling Operator, Sparse Coding

1 INTRODUCTION

Bag-of-Words [1], [2] (BoW) is a popular approach which transforms local image descriptors [3], [4], [5] into image representations that are used in retrieval and classification. To date, a number of its variants have been developed and reported to produce state-of-the-art results: Kernel Codebook [6], [7], [8], [9] a.k.a. Soft Assignment and Visual Word Uncertainty, Approximate Locality-constrained Soft Assignment [10], [11], Sparse Coding [12], [13], Local Coordinate Coding [14], and Approximate Locality-constrained Linear Coding [15]. We refer to this group as standard BoW. A second group improving upon BoW approaches includes Super Vector Coding [16], Vector of Locally Aggregated Descriptors [17], Fisher Vector Encoding [18], [19], and Vector of Locally Aggregated Tensors [20]. The main hallmarks of this group, in contrast to standard BoW, are: i) encoding of descriptors relative to the centres of their clusters, ii) extraction of second-order statistics from mid-level features to complement the first-order cues, iii) pooling with Power Normalisation [21], [19] which counteracts so-called burstiness [22], [11].

Several evaluations of various BoW models [23], [24], [25], [26], [11] address multiple aspects of BoW. A recent review of coding schemes [24] includes Hard Assignment, Soft Assignment, Approximate Locality-constrained Linear Coding, Super Vector Coding, and Fisher Vector Encoding. The role of pooling operators has been studied in [25], [26], [11] which lead to improvements in object category recognition. A detailed comparison of BoW [11] shows that the choice of pooling for various coders affects the classification performance. All evaluations highlight that the second group of methods, e.g. Fisher Vector Encoding perform significantly better than the standard BoW approaches.

The pooling step in standard BoW aggregates only first-order occurrences of visual words in the mid-level features and max-pooling [13] is often combined with various coding schemes [7], [13], [14]. In this paper, we study the BoW model according to the coding and pooling techniques and present ideas that improve the performance. The analysis of First-, Second-, and Third-order Occurrence Pooling in the BoW model constitutes the main contribution of this work. In more detail:

1) We propose Higher-order Occurrence Pooling that aggregates co-occurrences rather than occurrences of visual words in mid-level features, which leads to more discriminative representation. It is also presented as a novel approach for fusion of various descriptor types.

2) We introduce a new descriptor termed residual in the context of higher-order occurrence pooling that improves the accuracy of mid-level features as well as a novel pooling operator based on Higher Order Singular Value Decomposition [27], [28] and Power Normalisation [21].

3) We present an evaluation of First-, Second-, and Third-order Occurrence Pooling combined with several coding schemes and produce state-of-the-
art BoW results on several standard benchmarks.
We outperform Fisher Vector Encoding [19], [29] (FV), Vector of Locally Aggregated Tensors [20] (VLAT), Spatial Pyramid Matching [13], [30], and coder-free Second-order Pooling from [31].

Our method is somewhat inspired by Vector of Locally Aggregated Tensors [20] (VLAT) that also models the co-occurrences of features. Note that VLAT differs from VLAD [17] by employing second-order statistics. In contrast to VLAT, our approach allows to incorporate arbitrary coders and pooling operators. It also differs from the recently proposed Second-order Pooling applied to the low-level descriptors for segmentation [31]. In contrast, we perform pooling on the mid-level features thus still preserve the benefits from the coding step in BoW. In addition, we propose and evaluate Higher-order Occurrence Pooling. 2D histogram representation [32] is another take on building rich statistics from the mid-level features. In contrast, our approach results from the analytical solution to the kernel linearisation problems.

Recently, a significant change in the approach and performance of image recognition has been observed. Methods based on Deep Neural Networks [33] have started to dominate the field and little attention is now given to the BoW model. This paper discusses the latest achievements in BoW and demonstrates the performance on standard benchmarks. It also compares the results from BoW techniques to DNN methods.

The reminder of this section first introduces the standard model of Bag-of-Words in section 1.1. The coders and pooling operators used are presented in sections 1.2 and 1.3. The rest of this paper is organised as follows. Section 2 describes the BoW with Higher-order Occurrence Pooling. Section 3 proposes a new fusion method for various descriptors based on Higher-order Occurrence Pooling. Section 4 discusses Third-order Occurrence Pooling on low-level descriptors. Section 5 presents our experimental evaluations.

1.1 Bag-of-Words Model

Let us denote low-level descriptor vectors, such as SIFT [3], as \(x_n \in \mathbb{R}^D\) such that \(n = 1,\ldots,N\), where \(N\) is the total descriptor cardinality for the entire image set \(I\), and \(D\) is the descriptor dimensionality. Given any image \(i \in I\), \(N_i\) denotes a set of its descriptor indices. We drop the superscript for simplicity and use \(N\). Therefore, \(\{x_n\}_{n \in N}\) denotes a set of descriptors for an image \(i \in I\). Next, we assume \(k = 1,\ldots,K\) visual appearance prototypes \(m_k \in \mathbb{R}^D\) a.k.a. visual vocabulary, words, centres, atoms, or anchors. We form a dictionary \(\mathcal{M} = \{m_k\}_{k=1}^K\), where \(\mathcal{M} \in \mathbb{R}^{D \times K}\) can also be seen as a matrix with its columns formed by visual words. This is illustrated in figure 1. Following the formalism of [25], [11], we express the standard BoW approaches as a combination of the mid-level coding and pooling steps, followed by the \(\ell_2\) norm normalisation:

\[
\phi_n = f(x_n, \mathcal{M}), \quad \forall n \in N
\]

\[
h_k = g(\{\phi_{kn}\}_{n \in N})
\]

\[
h = \hat{h}/\|\hat{h}\|_2
\]

A mapping function \(f : \mathbb{R}^D \rightarrow \mathbb{R}^K\) in Equation (1). e.g. Sparse Coding [12] embeds each descriptor \(x_n\) into the visual vocabulary space \(\mathcal{M}\) resulting in mid-level features \(\phi_n \in \mathbb{R}^K\). Pooling operation \(g : \mathbb{R}^{N \times K} \rightarrow \mathbb{R}\) in Equation (2), e.g. Average or Max-pooling, aggregates occurrences of visual words in mid-level features, and therefore in an image. It uses all coefficients \(\phi_{kn}\) from visual word \(m_k\) for image \(i\) to obtain \(k^{th}\) coefficient in vector \(\hat{h} \in \mathbb{R}^K\). Note that \(\phi_n\) denotes an \(n^{th}\) mid-level feature vector while \(\phi_{kn}\) denotes its \(k^{th}\) coefficient. This formulation can also be extended to pooling over cells of Spatial Pyramid Matching as in [11]. Finally, signature \(\hat{h}\) is normalised in Equation (3). Signatures \(h_i, h_j \in \mathbb{R}^K\) for \(i, j \in I\) form a linear kernel \(\text{Ker}_{ij} = \langle h_i \rangle^T \cdot h_j\) used by a classifier. This model of BoW assumes First-order Occurrence Pooling and often uses SC, LLC, and LcSA coders that are briefly presented below.

1.2 Mid-level Coders

The mid-level coders \(f\) are presented in this section. Note that in \(x_n\) and \(\phi_n\) we skip index \(n\) to avoid clutter.

Sparse Coding (SC) [12], [13] expresses each local descriptor \(x\) as a sparse linear combination of the visual words from \(\mathcal{M}\). The following problem is solved:

\[
\phi = \arg\min_{\phi} \|x - \mathcal{M}\phi\|_2^2 + \alpha\|\phi\|_1
\]

s. t. \(\tilde{\phi} \geq 0\)

A low number of non-zero coefficients in \(\phi\), referred to as sparsity, is induced with the \(\ell_1\) norm and adjusted by constant \(\alpha\). We impose a non-negative constraint on \(\phi\) for compatibility with Analytical pooling [26], [11].

Approximate Locality-constrained Linear Coding (LLC) [15] addresses the non-locality phenomenon that can occur in SC and is formulated as follows:

\[
\phi^* = \arg\min_{\phi} \|x - \mathcal{M}(x, l) \phi\|_2^2
\]

s. t. \(\tilde{\phi} \geq 0, \quad 1^T \tilde{\phi} = 1\)

Descriptor \(x\) is coded with its \(l\)-nearest visual words \(\mathcal{M}(x, l) \in \mathbb{R}^{D \times l}\) found in dictionary \(\mathcal{M}\). Constant \(l \ll K\) controls the locality of coding. Lastly, the resulting \(\phi^* \in \mathbb{R}^l\) is re-projected into the full length nullified mid-level feature \(\phi \in \mathbb{R}^K\).

Approximate Locality-constrained Soft Assignment (LcSA) [10], [7] is derived from Mixture of Gaussians [34] \(G(x; m_k, \sigma)\) with equal mixing weights.
The component membership probability is used as a coder:
\[
\phi_k = \begin{cases} 
\frac{G(x; m_k, \sigma_k)}{\sum_{m' \in \mathcal{M}(x,l)} G(x; m', \sigma_k)} & \text{if } m_k \in \mathcal{M}(x,l) \\
0 & \text{otherwise}
\end{cases}
\]
where \(\phi_k\) is computed from the \(l\)-nearest Gaussian components of \(x\) that are found in dictionary \(\mathcal{M}\).

**Fisher Vector Encoding (FV)** [18], [19] uses a Mixture of \(K\) Gaussians as a dictionary. It performs descriptor coding w.r.t. to Gaussian components \(G(w_k, m_k, \sigma_k)\) which are parametrised by mixing probability, mean, and on-diagonal standard deviation. The first- and second-order features \(\phi_k, \psi_k \in \mathbb{R}^D\) are:
\[
\phi_k = (x - m_k)/\sigma_k, \quad \psi_k = \phi_k^2 - 1
\]
Concatenation of per-cluster features \(\bar{\phi}_k \in \mathbb{R}^{2KD}\) forms the mid-level feature \(\bar{\phi} \in \mathbb{R}^{2KD}:\)
\[
\phi = [\phi_T, ..., \phi_K]^T, \quad \bar{\phi}_k = \frac{p(m_k|x, \theta)}{\sqrt{\sigma_k}} [\phi_k \psi_k / \sqrt{2}]
\]
where \(p\) and \(\theta\) are the component membership probabilities and parameters of GMM, respectively. Note that this formulation is compatible with equation (1) except that \(\phi = 2KD\) rather than \(K\) dimensional. Also, \(\phi\) contains second-order statistics unlike codes of SC, LLC, and LcSA.

**Vector of Locally Aggregated Tensors (VLAT)** [20] also has a coding step that yields the first- and second-order features \(\phi_k \in \mathbb{R}^D\) and \(\Psi_k \in \mathbb{R}^{D \times D}\) per cluster:
\[
\phi_k = x - m_k, \quad \Psi_k = \phi_k \phi_k^T - C_k
\]
In contrast to Vectors of Locally Aggregated Descriptors [17] that employs first-order occurrences, only the second-order matrices \(\Psi_k\) are used to form the mid-level features after normalisation with covariance matrices \(C_k\) of \(k\)-means clusters. Each \(\Psi_k\) is symmetric, thus the upper triangles and diagonals are extracted and unfolded by operator \(u\) to form vector \(\phi\):
\[
\phi = [u(\Psi_1)^T, ..., u(\Psi_K)^T]^T
\]
This formulation is also compatible with equation (1) except that \(\phi = K D (D + 1)/2\) dimensional.

### 1.3 Pooling Operators

In BoW, pooling operators aggregate occurrences of visual words represented by the coefficients of mid-level feature vectors. The typically used pooling operators are presented below.

**Average pooling** [2] counts the number of descriptor assignments to each cluster \(k\) or visual word \(m_k\) and normalises such counts by the number of descriptors in the image. It is used with SC, LLC, LcSA, FV, VLAT and is defined as:
\[
\hat{h}_k = \text{avg} \left( \{\phi_{kn}\}_{n \in N} \right) = \frac{1}{|N|} \sum_{n \in N} \phi_{kn}
\]
**Max-pooling** [13], [10], [25], [26] selects the largest value from \(|N|\) mid-level feature coefficients corresponding to visual word \(m_k\):
\[
\hat{h}_k = \max \left( \{\phi_{kn}\}_{n \in N} \right)
\]
To detect occurrences of visual words, Max-pooling is often combined with SC, LLC, and LcSA coders. It is not applicable to FV or VLAT, as their mid-level feature coefficients do not represent visual words.

**MaxExp** [26] represents a theoretical expectation of Max-pooling inspired by a statistical model. The mid-level feature coefficients for a given \(m_k\) are presumed to be drawn at random from Bernoulli distribution under the i.i.d. assumption. From binomial distribution, given exactly \(\bar{N} = |N|\) trials, the probability of at least one visual word \(m_k\) present in an image is:
\[
\hat{h}_k = 1 - \left( 1 - h_k^* \right)^{\bar{N}}, \quad h_k^* = \text{avg} \left( \{\phi_{kn}\}_{n \in N} \right)
\]
This operator aggregates \(\bar{N}\) independent features but number \(\bar{N} \leq |N|\) has to be found by cross-validation. MaxExp is typically used with SC, LLC, and LcSA as constraint \(0 \leq h_k^* \leq 1\) does not hold for FV or VLAT.

**Power Normalisation** a.k.a. Gamma [21], [19], [22] approximates the statistical model of MaxExp as shown in [11] and is used by SC, LLC, LcSA, FV, VLAT:
\[
\hat{h}_k = \text{sgn} (h_k^*) |h_k^*|^\gamma, \quad h_k^* = \text{avg} \left( \{\phi_{kn}\}_{n \in N} \right)
\]
The influence of statistical dependence between features is controlled by \(0 < \gamma \leq 1\).

### @n pooling [11] was designed to suppress the low values of mid-level feature coefficients that are considered a noise in SC, LLC, and LcSA. This operator is based on MaxExp and considered a trade-off between Max-pooling and Analytical pooling [26], [11]:
\[
\hat{h}_k = 1 - (1 - h_k^*)^{\bar{N}}, \quad h_k^* = \text{avg srt} \left( \{\phi_{kn}\}_{n \in N, @n} \right)
\]
The @n largest mid-level features are selected by partial sort algorithm srt and averaged by avg. \(\bar{N}\) is the number of averaged features such that \(1 \leq \bar{N} \leq @n \leq |N|\). The mid-level feature coefficients for any \(m_k\) are presumed to be drawn from a Bernoulli distribution under the i.i.d. assumption. Given exactly \(\bar{N} = @n\) trials, equation (15) yields the probability of at least one visual word \(m_k\) present among the @n largest mid-level feature coefficients. Assuming that large \(\phi_{kn}\) correspond to visual words \(m_k\), @n cuts off small \(\phi_{kn}\) that originate from noise. This does not hold for FV or VLAT as their small \(\phi_{kn}\) are not considered noise.

### 2 Bag-of-Words with Higher-Order Occurrence Pooling

In this section, we introduce the Higher-Order Occurrence Pooling for BoW with its derivation in sections 2.1 and 2.2.
summing over all pairs of mid-level features formed from a pair of images, iii) normalising sums by the feature counts and, iv) normalising the resulting kernel. First, we define Minor Polynomial Kernel:

$$k_r (\hat{\phi}, \bar{\phi}) = \left( \beta \phi^T \bar{\phi} + \lambda \right)^r$$  \hspace{1cm} (18)

We chose \(\beta = 1\) and \(\lambda = 0\), while \(r \geq 1\) denotes the polynomial degree and the order of occurrence pooling. Equation (18) reduces to the dot product \(k_r (\phi, \bar{\phi}) = \langle \phi, \bar{\phi} \rangle^r\) of a pair of mid-level features. The mid-level features result from \(\mathcal{N}\) and \(\mathcal{N}'\) descriptors in two images. We assume \(\phi\) and \(\bar{\phi}\) are the \(\ell_2\) normalised. We define a kernel function between two sets \(\Phi = \{ \phi_n \}_{n \in \mathcal{N}}\) and \(\bar{\Phi} = \{ \bar{\phi}_n \}_{n \in \mathcal{N}'}:\n
$$\text{Ker}(\Phi, \bar{\Phi}) = \frac{1}{|\mathcal{N}|} \sum_{n \in \mathcal{N}} \frac{1}{|\mathcal{N}'|} \sum_{n' \in \mathcal{N}'} \langle \phi_n, \bar{\phi}_{n'} \rangle^r$$

$$= \frac{1}{|\mathcal{N}|} \sum_{n \in \mathcal{N}} \frac{1}{|\mathcal{N}'|} \sum_{n' \in \mathcal{N}'} \left( \sum_{k=1}^{K} \phi_{kn} \bar{\phi}_{kn'} \right)^r \hspace{1cm} (19)$$

The rightmost summation in equation (19) can be expressed as a dot product of two self-tensor products of order \(r\):

$$\left( \sum_{k=1}^{K} \phi_{kn} \bar{\phi}_{kn'} \right)^r = \sum_{k^{(1)}=1}^{K} \cdots \sum_{k^{(r)}=1}^{K} \phi_{k^{(1)}n} \bar{\phi}_{k^{(1)}n'} \cdots \phi_{k^{(r)}n} \bar{\phi}_{k^{(r)}n'}$$

$$= \langle u_1 (\otimes \phi_n), u_1 (\otimes \bar{\phi}_n) \rangle$$  \hspace{1cm} (20)

By combining equations (19) and (20) we have:

$$\text{Ker}(\Phi, \bar{\Phi}) = \frac{1}{|\mathcal{N}|} \sum_{n \in \mathcal{N}} \frac{1}{|\mathcal{N}'|} \sum_{n' \in \mathcal{N}'} \langle u_1 (\otimes \phi_n), u_1 (\otimes \bar{\phi}_{n'}) \rangle$$

$$= \frac{1}{|\mathcal{N}|} \sum_{n \in \mathcal{N}} \frac{1}{|\mathcal{N}'|} \sum_{n' \in \mathcal{N}'} u_1 (\otimes \phi_n), u_1 (\otimes \bar{\phi}_{n'})$$

$$= \left( \frac{\text{avg} \left[ u_1 (\otimes \phi_n) \right], \text{avg} \left[ u_1 (\otimes \bar{\phi}_n) \right] }{n \in \mathcal{N}} \right)$$

$$= \text{Ker}(\Phi, \bar{\Phi})$$  \hspace{1cm} (21)

Finally, \(\text{Ker}(\Phi, \bar{\Phi})\) is normalised to ensure self-similarity \(\text{Ker}(\Phi, \bar{\Phi}) = \text{Ker}(\bar{\Phi}, \Phi) = 1\), by:

$$\text{Ker}(\Phi, \bar{\Phi}) := \frac{\text{Ker}(\Phi, \bar{\Phi})}{\sqrt{\text{Ker}(\Phi, \bar{\Phi}) \text{Ker}(\Phi, \Phi)}}$$  \hspace{1cm} (22)

The model derived in equation (21) is the BoW defined in equations (1), (16), and (17).

2.2 Beyond Average Pooling of Higher-order Occurrences

This section provides an extension of the proposed Higher-order Occurrence Pooling such that it can be combined with Max-pooling, which was reported to outperform the Average pooling in visual recognition [13], [26], [11]. We note that Average pooling counts all occurrences of a given visual word in an image, thus it quantifies areas spanned by repetitive
patterns while Max-pooling only detects the presence of a visual word in an image. Max-pooling was shown to be a lower bound of the likelihood of at least one visual word \( m_k \) being present in an image \([10]\).

First, we define \( \max \) operators on mid-level features: i) \( \max_{n \in N} \phi_n = \max \{ \{ \phi_n \}_{n \in N} \) and ii) \( \max_{n \in N} \Phi_n \) as a vector formed from element-wise \( \max \{ \{ \phi_1 \}_{n \in N} \), \( \max \{ \{ \phi_2 \}_{n \in N} \), ..., \( \max \{ \{ \phi_K \}_{n \in N} \). Note that \( \Phi = \{ \phi_n \}_{n \in N} \) and \( \Phi = \{ \phi_n \}_{n \in N} \) are two sets of mid-level features formed by \( N \) and \( \bar{N} \) descriptors from a pair of images. BoW with Max-pooling and Polynomial Kernel of degree \( r \) is given in equation (23) which is then expanded in equation (24) and simplified to a dot product between two vectors in equation (25) such that it forms a linear kernel. A lower bound of this kernel is proposed in equation (26), which represents Higher-order Occurrence Pooling with the Max-pooling operator. We further express it as a dot product between two vectors in equation (27).

\[
\begin{align*}
Ker(\Phi, \bar{\Phi}) &= \langle \hat{h}, \bar{h} \rangle^r, \text{ and } \\
\hat{h}_k &= \max_{n \in N} \{ \{ \phi_{kn} \}_{n \in N} \} \\
\bar{h}_k &= \max_{n \in \bar{N}} \{ \{ \phi_{kn} \}_{n \in \bar{N}} \}
\end{align*}
\]

(23)

\[
\begin{align*}
K &= \sum_{k^{(1)}} \cdots \sum_{k^{(r)}} \max_{n \in N} \{ \{ \phi(k^{(1)} \cdots \phi(k^{(r)} \}_n \} \cdot \\
\max_{n \in \bar{N}} \{ \{ \phi(k^{(1)} \cdots \phi(k^{(r)} \}_n \} \\
= \langle u, [\otimes_{r} \max_{n \in N} \{ \phi_n \}] \rangle, \langle u, [\otimes_{r} \max_{n \in \bar{N}} \{ \phi_n \}] \rangle \\
\geq \sum_{k^{(1)}} \cdots \sum_{k^{(r)}} \max_{n \in N} \{ \{ \phi(k^{(1)} \cdots \phi(k^{(r)} \}_n \} \cdot \\
\max_{n \in \bar{N}} \{ \{ \phi(k^{(1)} \cdots \phi(k^{(r)} \}_n \} \\
= \max_{n \in N} \{ \{ \otimes_{r} \phi_n \} \}, \max_{n \in \bar{N}} \{ \{ \otimes_{r} \phi_n \} \}
\end{align*}
\]

(27)

The formulation with Average pooling, which preserves bi-linearity in equation (21), is convenient for the linearisation but breaking bi-linearity leads to improvements as demonstrated in \([20]\). Equation (27) introduces max-pooling that breaks bi-linearity and enables the use of other suitable operators for Higher-order Occurrence Pooling such as the \( \otimes_n \) operator.

An interesting probabilistic difference between the BoW models in equations (25) and (27) can be shown. We first consider Max-pooling in regular BoW with a linear kernel. If mid-level feature coefficients \( \phi_{kn} \) are drawn from a feature distribution under the i.i.d. assumption given a visual word \( m_k \), the likelihood of at least one visual word \( m_k \) being present in an image \([10]\) is an upper bound of Max-pooling:

\[
1 - \prod_{n \in N} (1 - \phi_{kn}) \geq \max \{ \{ \phi_{kn} \}_{n \in N} \}
\]

(28)

Fig. 2. Uncertainty in Max-pooling. Mid-level feature coefficients \( \phi_1 \) and \( \phi_2 \) are produced for descriptors \( 1 \leq x \leq 2 \) given visual words \( m_1 = 1 \) and \( m_2 = 2 \). (a) First-order Occurrence Pooling results in the pooling uncertainty \( u \) (the grey area). See text for explanations. (b) Second-order statistics produce co-occurrence component \( (\phi_1 \phi_2)^{0.5} \) that has a maximum for \( x \) indicated by the dashed stem. This component limits the pooling uncertainty. The square root is applied to preserve the linear slopes, e.g. \( (\phi_1 \phi_2)^{0.5} = \phi_1 \).

We now derive upper bounds of Max-pooling for the BoW models in equations (25) and (27). We denote the image signature from equation (25) as tensor \( T = \otimes_{r} \max \{ \phi_n \} \in \mathbb{R}^{K^r} = \{ T_{k^{(1)}}, ..., T_{k^{(r)}} \} \). Coefficient-wise, it is:

\[
T_{k^{(1)}}, ..., k^{(r)} = \prod_{s=1}^{r} \max \{ \{ \phi_{s(n)} \}_{n \in N} \}
\]

(29)

Each coefficient of an image signature obtained with Max-pooling and Polynomial Kernel is upper bounded by the probability of visual words \( m_{k^{(1)}}, ..., m_{k^{(r)}} \) jointly occurring at least once after pooling:

\[
T_{k^{(1)}}, ..., k^{(r)} \leq \prod_{s=1}^{r} \left( 1 - \prod_{n \in N} (1 - \phi_{s(n)}) \right)
\]

(30)

The image signature from equation (27) forms tensor \( T' = \max \{ \otimes_{r} \phi_n \} \in \mathbb{R}^{K^r} \). Coefficient-wise, this is:

\[
T'_{k^{(1)}}, ..., k^{(r)} = \max \{ \{ \prod_{s=1}^{r} \phi_{s(n)} \}_{n \in N} \}
\]

(31)

Again, we note that each coefficient of an image signature is upper bounded by the probability of visual words \( m_{k^{(1)}}, ..., m_{k^{(r)}} \) jointly occurring in at least one mid-level feature \( \phi_n \) before pooling:

\[
T'_{k^{(1)}}, ..., k^{(r)} \leq \prod_{n \in N} \left( 1 - \prod_{s=1}^{r} \phi_{s(n)} \right)
\]

(32)

Unlike the joint occurrence after pooling in equation (30), the joint occurrence of visual words computed in equation (32) before pooling can be interpreted as a new auxiliary element in the visual vocabulary.

Intuitive illustration. We argue that the joint occurrence of visual words in the mid-level features benefits Max-pooling (and other related operators) by limiting the uncertainty about the presence of descriptors. Figure 2 illustrates the mid-level coefficients given one dimensional visual words \( m_1 = 1 \) and \( m_2 = 2 \). It shows two linear slopes representing coding values \( \phi_1 \) and \( \phi_2 \) for any descriptor from range \( 1 \leq x_n \leq 2 \).
If all \( x_n = 1.5 \) then \( \phi_{1n} = \phi_{2n} = 0.5 \) for all \( n \). Applying Max-pooling would result in \( \max(\{\phi_{1n}\}_{n \in \mathcal{N}}) = \max(\{\phi_{2n}\}_{n \in \mathcal{N}}) = 0.5 \). This signature uniquely indicates the presence of \( x_n = 1.5 \), therefore uncertainty of \( x_n \) location is \( u = 0 \). However, if \( x_n \) have different values from the given range, due to Max-pooling, the two largest coefficients \( \phi_{1n} \) and \( \phi_{2n} \) for the two descriptors closest to \( m_1 \) and \( m_2 \) would mask the presence of other descriptors, i.e. the mid-level signature would not contain any information about other descriptors. Thus, as \( \max(\{\phi_{1n}\}_{n \in \mathcal{N}}) \rightarrow 1 \) and \( \max(\{\phi_{2n}\}_{n \in \mathcal{N}}) \rightarrow 1 \), the uncertainty in location of other descriptors \( x_n \) is \( u \rightarrow 1 \). We argue that the role of pooling is to aggregate the mid-level features into a signature that preserves information about all descriptors.

Figure 2(b) extends the above example with the second-order statistics which, in addition to \( \phi_1 \) and \( \phi_2 \), introduces \( \phi_3 \) and \( \phi_4 \). Its maximum occurs for descriptor \( x_n = 1.5 \). We applied the square root \( \sqrt{\phi_1 \phi_2} \) to preserve the linear slopes of \( \phi_1 \) and \( \phi_2 \) in the plot. Note that Max-pooling is applied to the individual \( \max(\{\phi_{1n}\}_{n \in \mathcal{N}}) \) and \( \max(\{\phi_{2n}\}_{n \in \mathcal{N}}) \) as well as to the joint term \( \max(\{\phi_{1n}\phi_{2n}\}_{n \in \mathcal{N}}) \). This term indicates the presence of descriptors in the mid-range between \( m_1 \) and \( m_2 \). The second-order statistics limit the uncertainty of Max-pooling such that \( u_1 + u_2 \leq u \) thus increase the resolution of the visual dictionary.

3 Mid-level Feature Fusion with Higher-Order Occurrence Pooling
Shape, texture and colour cues are often combined for object category recognition [5], [35], [19], [36], [37], [38] and visual concept detection [39], [40], [41], [42], [11]. Some approaches employ so-called early fusion on the low-level descriptors [5], [35], [43]. Other methods apply coding and pooling on various modalities first, followed by so-called late fusion of multiple kernels [36], [37], [38], [35], [41].

We first formalise the early and late fusions, which are used as a baseline for comparisons to our fusion method based on Higher-order Occurrence Pooling. It captures the co-occurrences of visual words in each mid-level feature as shown in equation (27) of section 2.2. This is extended here to multiple descriptor types via linearisation of Minor Polynomial Kernel.

3.1 Early and Late Fusion
Early fusion is typically referred to when different types of low-level descriptors are concatenated. Such a fusion of various descriptors with their spatial coordinates was introduced in [43] as Spatial Coordinate Coding. It was presented as a low dimensional alternative to Spatial Pyramid Matching [30]. A similar fusion was used by others, e.g. in Joint Sparse Coding [44].

Low-level descriptor vector \( x \) and dictionary \( \mathcal{M} \) can be formed by concatenating \( Q \) descriptor types:

\[
x = \left[ \sqrt{\beta(1)} x^{(1)} T , \ldots , \sqrt{\beta(Q)} x^{(Q)} T \right] T
\]

\[
\mathcal{M} = \left[ \sqrt{\beta(1)} \mathcal{M}^{(1)} , \ldots , \sqrt{\beta(Q)} \mathcal{M}^{(Q)} \right] T
\]  (33)

Weights \( \beta^{(1)}, \ldots , \beta^{(Q)} \) determine the contribution of descriptor types \( x^{(1)}, \ldots , x^{(Q)} \) given dictionaries \( \mathcal{M}^{(1)}, \ldots , \mathcal{M}^{(Q)} \) and are typically learnt from the data.

Spatial Coordinate Coding (SCC) [43] is an example of early fusion which extends descriptors \( x \) with their spatial locations \( x_s = [c_x, c_y, h] T \) normalised by the image width and height. Thus \( x := [\sqrt{\beta} x_s^T , \sqrt{1 - \beta} x_s^T , \sqrt{\beta} x_s^T]^T \).

Late Fusion [36], [42] of multiple descriptor types is performed by independently coding and pooling each type and combining their kernels:

\[
Ker_{ij} = \sum_{q=1}^{Q} \beta^{(q)} Ker_{iq}^{(q)}
\]  (34)

There are various approaches to learn kernel weights \( \beta^{(q)} \) [38], [35], [41]. However, for a small number of kernels, cross-validation can be used.

3.2 Linearisation of Minor Polynomial Kernel
The kernel linearisation for multiple descriptor types follows the approach detailed in section 2.1. We define Minor Polynomial Kernel:

\[
ker \left( \{ (\phi^{(q)}, \bar{\phi}^{(q)}) \} \right) \approx \left( \sum_{q=1}^{Q} \beta^{(q)} \phi^{(q)} T \bar{\phi}^{(q)} + \lambda \right)^r
\]  (35)

There is one pair of mid-level features \( (\phi^{(n)}, \bar{\phi}^{(n)}) \) per each descriptor type \( q \) with weights \( \beta^{(q)} \) adjusting the contributions. Similarly to equation (19), we obtain:

\[
ker \left( \{ (\phi^{(n)}, \bar{\phi}^{(n)}) \} \right) \approx \left( \sum_{q=1}^{Q} \beta^{(q)} \phi^{(q)} T \bar{\phi}^{(q)} \right)^r
\]  (36)

The kernel function is evaluated between two images represented by two sets of mid-level features \( \Phi = \{ (\phi^{(n)}_n)_{n \in \mathcal{N}} \} \) and \( \bar{\Phi} = \{ (\bar{\phi}^{(n)}_n)_{n \in \mathcal{N}} \} \) from \( \mathcal{N} \) and \( \bar{\mathcal{N}} \) descriptors and \( Q \) modalities:

\[
Ker(\Phi, \bar{\Phi}) = \frac{1}{|\mathcal{N}|} \sum_{n \in \mathcal{N}} \frac{1}{|\bar{\mathcal{N}}|} \sum_{\bar{n} \in \bar{\mathcal{N}}} \left( \sum_{q=1}^{Q} \beta^{(q)} (\phi^{(q)}_n, \bar{\phi}^{(q)}_{\bar{n}}) \right)^r
\]

\[
= \frac{1}{|\mathcal{N}|} \sum_{n \in \mathcal{N}} \frac{1}{|\bar{\mathcal{N}}|} \sum_{\bar{n} \in \bar{\mathcal{N}}} \left( \sum_{q=1}^{Q} \beta^{(q)} (\phi^{(q)}_n, \bar{\phi}^{(q)}_{\bar{n}}) \right)^r
\]  (37)

Bi-modal Second-order Occurrence Pooling is first derived by linearising the above kernel for \( Q = 2 \) (two coders) and \( r = 2 \) (second-order). We denote \( \beta^{(1)} = \beta^{(2)} \).
Combining these terms with equation (38), we obtain three dot products:

\[
\left( K \sum_{k=1}^{K} \phi_{kn}^{(1)} \phi_{kn}^{(1)} + (1-\beta) \sum_{k=1}^{K} \phi_{kn}^{(2)} \phi_{kn}^{(2)} \right)^{2} \]

and \( \beta(2) = 1-\beta \). Thus, by expanding the square term in equation (38), we obtain three dot products:

\[
\beta^{2} \left( u_{:}^{(1)} \phi_{n}^{(1)} \right)^{T} \left( u_{:}^{(1)} \phi_{n}^{(1)} \right) + 2\beta(1-\beta) \left( u_{:}^{(1)} \phi_{n}^{(1)} \right)^{T} \left( u_{:}^{(2)} \phi_{n}^{(2)} \right) + (1-\beta)^{2} \left( u_{:}^{(2)} \phi_{n}^{(2)} \right)^{T} \left( u_{:}^{(2)} \phi_{n}^{(2)} \right)
\]

Combining these terms with equation (37) yields:

\[
\text{Ker}(\Phi, \Phi) = \beta^{2} \left( \text{avg}_{n \in N} \left[ u_{:}^{(1)} \phi_{n}^{(1)} \right] \right)^{2} + 2\beta(1-\beta) \left( \text{avg}_{n \in N} \left[ u_{:}^{(1)} \phi_{n}^{(1)} \right] \right) \left( \text{avg}_{n \in N} \left[ u_{:}^{(2)} \phi_{n}^{(2)} \right] \right) + (1-\beta)^{2} \left( \text{avg}_{n \in N} \left[ u_{:}^{(2)} \phi_{n}^{(2)} \right] \right)^{2}
\]

Note that the first and the last terms represent Second-order Occurrence Pooling for independent coders \( q = 1 \) and \( q = 2 \) and correspond to equation (21) in section 2.1. However, the middle dot product represents the cross-term that captures additional information in form of the co-occurrences between visual words of mid-level features from two coders. Figure 3 illustrates this model.

Bi-modal Higher-order Occurrence Pooling can also be derived by expanding Minor Polynomial Kernel in equation (36). For order \( r \geq 2 \) and two coders \( Q = 2 \), by substituting \( a = \left\langle \phi^{(1)}, \bar{\phi}^{(1)} \right\rangle \) and \( b = \left\langle \phi^{(2)}, \bar{\phi}^{(2)} \right\rangle \), one can expand Minor Polynomial Kernel in equation (36) using Binomial theorem:

\[
\left( \beta a + (1-\beta)b \right)^{r} = \sum_{s=0}^{r} \binom{r}{s} \beta^{r-s} (1-\beta)^{s} \]

The derivations follow the same steps as for Bi-modal Second-order Occurrence Pooling. We skip that for clarity and define Bag-of-Words with Bi-modal Higher-order Occurrence Pooling:

\[
\phi_{n}^{(1)} = f(1) \left( x_{n}^{(1)}, \mathcal{M}^{(1)} \right) \quad \forall n \in N \quad (44)
\]

\[
\phi_{n}^{(2)} = f(2) \left( x_{n}^{(2)}, \mathcal{M}^{(2)} \right) \quad (45)
\]

\[
\psi_{n}^{s} = u_{:} \left[ \left( \otimes_{r-s} \phi_{n}^{(1)} \right) \left( \otimes_{s} \phi_{n}^{(2)} \right) \right], \quad s = 0, ..., r
\]

Equations (44) and (45) follow the terminology from equations (1) and (16) and represent the coding step for two coders. The coders can be of different types and their dictionary sizes \( K^{(1)} \) and \( K^{(2)} \) may differ. Equation (45) represents the joint occurrence of visual words in \( \phi_{n}^{(1)} \) or \( \phi_{n}^{(2)} \), or the cross-modal joint occurrence of visual words per mid-level pair \( (\phi_{n}^{(1)}, \phi_{n}^{(2)}) \). It results from an expansion of Minor Polynomial Kernel in equation (36) according to Binomial theorem in a similar way to equations (38-41). The dimensionality of \( \psi_{s} \) after removing repeated coefficients and unfolding is \( K^{(r,s)} = K^{(r,s)}K^{(s)} \). Equation (46) represents pooling that aggregates the joint occurrences or the cross-modal joint occurrences of visual words. Function \( g^{(s)} : \mathbb{R}^{|N|} \rightarrow \mathbb{R} \) uses the \( k \)th joint occurrence to produce the \( k \)th coefficient in vector \( \hat{h}^{s} \in \mathbb{R}^{K^{(r,s)}} \). The weighting factors preceding \( g^{(s)} \) result from Binomial expansion (cf. equation (43)). Equation (47) concatenates and normalises the joint occurrence statistics.

Multi-modal Higher-order Occurrence Pooling can be derived in the same way by using Multinomial instead of Binomial theorem and expanding Minor Polynomial Kernel in equation (36). The fusion can be performed by concatenating the mid-level features from \( Q \) coders:

\[
\phi_{n} = \left[ \sqrt{\beta(1)} \phi_{n}^{(1)} \phi_{n}^{(1)}, \sqrt{\beta(2)} \phi_{n}^{(2)} \phi_{n}^{(2)}, ..., \sqrt{\beta(Q)} \phi_{n}^{(Q)} \phi_{n}^{(Q)} \right]^{T}
\]

Thus, mid-level features \( \phi_{n} \) form tensors (cf. equation (16)) and can form Bi- or Multi-modal Second- and Higher-order Occurrence Pooling.

### 3.3 Residual Descriptors

In this section, we introduce an approach based on Bi-modal Second-order Occurrence Pooling that further improves the accuracy of coding. Various coding approaches such as SC and LLC optimise a trade-off between the quantisation loss (defined below) and a chosen regularisation penalty, e.g. sparsity or locality as in equations (4) and (5). The quality of quantisation in coders can be measured based on the theory of Linear Coordinate Coding [14]. The linear approximation of descriptor \( x \) given dictionary \( \mathcal{M} \) and coder \( f \) that produces mid-level feature \( \phi = \mathcal{M}f(x) = \mathcal{M}\phi. \)
The quantisation loss a.k.a quantisation error is then defined as:

$$\xi^2 = \|x - \hat{x}\|^2. \quad (49)$$

However, $\xi^2$ quantifies only its magnitude. 

**Residual Descriptor (RD)** is therefore defined as:

$$\xi = x - M\phi \quad (50)$$

and illustrated in figure 4. Descriptors $x \in [-3, 3]^2$ are coded with three atoms $m_1 = [0, 3]^T$, $m_2 = [-2, -2]^T$, and $m_3 = [2, -2]^T$ by SC and LcSA. The mid-level features $\phi$ are projected back to the descriptor space: $\hat{x} = M\phi$. The resulting quantisation loss, i.e., RD, are visualised by displacements between each descriptor $x$ and its approximation $\hat{x}$. Figure 4(a) shows low quantisation loss for SC with regularisation $\alpha = 1$ and figure 4(b) shows large quantisation errors for LcSA due to low $l = 2$.

To better represent the original $x$, both the mid-level feature $\phi$ and the Residual Descriptor $\xi$ are incorporated into the signature as two different descriptors:

$$\phi_n^{(1)} = f(x_n, M), \quad \phi_n^{(2)} = x_n - M\phi_n^{(1)} \quad (51)$$

With this formulation, the cross-term of the Bi-modal Second-order Occurrence Pooling can capture co-occurrences between mid-level feature $\phi^{(1)}$ encoding original descriptor $x$ and the corresponding residual error $\phi_n^{(2)}$. Thus, it associates the error with the descriptor and improves the coding accuracy.

### 4 Pooling Low-level Descriptors

Recently, a coder-free approach was proposed in [31] in the context of semantic segmentation. This method avoids mid-level coding and employs the autocorrelation matrix formed by Average pooling of the outer products of local image descriptors. The matrix is then normalised with the log operator. We go beyond the second-order and generalise this approach to Higher-order Occurrence Pooling as well as propose a two-stage normalisation based on eigenvalue decomposition and Power Normalisation.

**Eigenvalue Power Normalisation.** Corrections such as Power Normalisation (cf. section 1.3) are known to improve the Average pooling [21], [19], [22]. This is related to the problem of burstiness which was defined in [22] as “the property that a given visual element appears more times in an image than a statistically independent model would predict”. The Analytical pooling operators [26], [11] have been advocated as a remedy to the burstiness phenomenon. They act similarly to the MaxExp operator (cf. section 1.3) which approximates the probability of at least one particular visual word being present in an image. They are applied to each coefficient in mid-level features, which are assumed to be i.i.d., therefore they can also be interpreted as whitening of the i.i.d. coefficients. We argue that the i.i.d. assumption does not always hold in real images. For instance, local descriptors extracted from repetitive texture patterns frequently co-occur and their coefficients are correlated. Thus, the burstiness of these descriptors can be addressed effectively by decorrelating along the principal components of the signal rather than coefficient-wise normalisations.

We thus propose to perform Power Normalisation, MaxExp, or a similar correction on the eigenvalues of the higher-order tensor coefficients. Figures 5 (a) and (b) show the difference between the eigenvalue (ePN) and coefficient-wise (PN) Power Normalisation. Autocorrelation matrix $H$ was built from correlated 2D features $\phi$. The principal components of $H_0.4$, $H_0.1$, $H^{0.4}$, and $H^{0.1}$ show the data being whitened with ePN to a significant extent. On the contrary, element-wise Power Normalisation (PN) fails to whiten the correlated data.

**Higher-order Pooling with eigenvalue Power Normalisation (ePN).** Second-order Occurrence Pooling can be performed by applying Power Normalisation to the eigenvalues of the autocorrelation matrix. For the second-order matrix, we use Singular Value Decomposition. For the Third-order Occurrence Pooling, we employ Higher Order Singular Value Decomposition (HOSVD) [27], [28]. Power Normalisation is
then performed on the eigenvalues from so-called core tensor and the autocorrelation tensor is reassembled:
\[
\phi_n = x_n, \quad \forall n \in N \tag{52}
\]
\[
H = \operatorname{avg}_{n \in N} (\Phi_n), \quad \Phi_n = \otimes_r \phi_n \tag{53}
\]
\[
(E; A_1, \ldots, A_r) = \operatorname{hosvd}(H) \tag{54}
\]
\[
\tilde{E} = \operatorname{sgn}(E) \frac{E}{\|E\|} \tag{55}
\]
\[
\tilde{H} = \tilde{E} \times A_1 \cdots \times_r A_r \tag{56}
\]
\[
\hat{h} = \operatorname{sgn}(h*) |h*|^\gamma, \quad h* = u_r(\tilde{H}) \tag{57}
\]

Coder-free image signatures are represented in equation (52), however, to reduce their size, we apply PCA \( \phi_n = \operatorname{pcaproj}(x_n) \) and obtain \( \phi_n \in \mathbb{R}^K \). We investigate three variants of these features based on the use of spatial information: i) no spatial information, ii) appended on the descriptor level, i.e. Spatial Coordinate Coding, iii) appended according to equation (48) of spatial information: i) no spatial information, ii) Higher-order Occurrence Pooling according to equation (52), however, to reduce their size, we apply PCA and coefficient-wise correction such as Power Normalisation, respectively. In Equation (54), HOSVD denoted by operator \( \operatorname{hosvd} : \mathbb{R}^{K'} \rightarrow (\mathbb{R}^K)^{K \times 1} \) decomposes the higher-order autocorrelation tensor \( H \in \mathbb{R}^{K'} \) (an \( r \)-th-order equivalent of the autocorrelation matrix) is computed by averaging over tensors \( \Phi_n \in \mathbb{R}^{K'} \).

Equations (54–56) and (57) represent two stage pooling with eigenvalue- and coefficient-wise corrections such as Power Normalisation, respectively. In Equation (54), \( \operatorname{hosvd} \) decomposes the higher-order autocorrelation tensor \( H \) into core tensor \( E \) of eigenvalues and orthonormal factor matrices \( A_1, \ldots, A_r \in \mathbb{R}^{K \times 1} \), which can be interpreted as the principal components in \( r \) modes. Element-wise corrections are then applied to eigenvectors \( E \) by Power Normalisation (cf. equation (55)). The higher-order autocorrelation tensor \( \tilde{H} \in \mathbb{R}^{K'} \) is reassembled in equation (56) by \( r \)-mode product \( x_r \) (detailed in [28]) of normalised tensor \( \tilde{E} \) and \( A_1, \ldots, A_r \). Operator \( v* \) is used in equation (57) to remove the redundant coefficients from symmetric tensor \( \tilde{H} \) and coefficient-wise correction is applied to \( h* \). Coefficient-wise correction, i.e. MaxExp from equation (13), can be also applied here. Note that the \( \ell_2 \) normalisation (cf. equation (3)) is always applied at the end.

5 EXPERIMENTAL SECTION

We first introduce our experimental settings in section 5.1. First-, Second-, and Third-order Occurrence Pooling are compared to FV and VLAT in section 5.2. Various descriptor fusion techniques with Higher-order Occurrence Pooling are evaluated in section 5.3. Higher-order Occurrence Pooling variants for the low-level descriptors are compared in section 5.4 and other pooling techniques are evaluated in section 5.5.

5.1 Experimental Settings

Eight widely used image recognition benchmarks were used in our experiments. The datasets, descriptor parameters, various experimental details and state-of-the-art results are summarised in table 1. Other settings are discussed below.

Descriptors. Opponent SIFT was extracted on dense grids. Either grey scale only (128D) or grey and colour components (128D+144D) were used as detailed in table 1. PCA was applied to reduce descriptor dimensionality to 80D for the grey and 120D opponent components in FV and VLAT.

Spatial bias. Spatial relations in images were exploited mainly by Spatial Coordinate Coding [43] described in section 3.1. Spatial Pyramid Matching (SPM) [30] and Dominant Angle Pyramid Matching (DoPM) [43] were additionally used for comparison to the standard BoW with first-order occurrences. Multiple spatial grids such as 1x1, 1x3, 3x1 or 1x1, 2x2, 3x3, and 4x4 were used in SPM [30]. DoPM [43] was used to exploit dominant gradient orientations with 5 quantisation levels of 1, 3, 6, 9, and 12 angular grids. BoW with first-order occurrences employed either SCC or SPM with/without DoPM as detailed later. As recommended in [29], we use FV and VLAT with SCC rather than SPM.

Dictionaries. Online Dictionary Learning [58] was used to train dictionaries for Sparse Coding. Dictionary learning proposed in [15] was shown to outperform k-means, we therefore used it for Locality-constrained Linear Coding and adapted it to work with Approximate Locality-constrained Soft Assignment. Dictionary size was varied from 4K to 40K for First-, 300 to 1600 for Second-, and 100 to 200 for Third-order Occurrence Pooling. Fisher Vector Encoding [18] and Vector of Locally Aggregated Tensors [20] were used in comparisons, GMM and k-means dictionaries with 64 to 4096 and 64 to 512 atoms were employed, respectively.

Coding and Pooling. Unless stated otherwise, Sparse Coding SC and our \( \ell^0 \) pooling operator were used in the experiments. Additional results for Max-pooling, MaxExp, Power Normalisation and the proposed eigenvalue based normalisation are also provided. FV and VLAT were combined with Power Normalisation only as other operators are not directly applicable.

Classifiers. Kernel Discriminant Analysis [59] with linear kernels \( \operatorname{Ker}_{ij} = (h_j)^T \cdot h_i \) was applied in all experiments (unless stated otherwise), where \( h_i, h_j \in \mathbb{R}^K \) are signatures for images \( i \) and \( j \). Mean Accuracy (acc.) and Mean Average Precision (mAP) are used in evaluations (see table 1).

5.2 Bag-of-Words with First-, Second-, and Third-order Occurrence Pooling

This section compares the performance of the proposed Higher-order Occurrence Polling to the state—
of-the-art approaches, e.g. FV and VLAT, on PascalVOC07 and Caltech101 benchmarks. The results are reported for BoW (cf. section 2) with Sparse Coding of grey scale SIFT and occurrence orders \( r = 1, 2, \) and 3. Note that the BoW model with \( r = 1 \) is equivalent to the standard BoW (cf. section 1.1).

Figure 6(a) compares performance for various orders \( r \). BoW with Second-order Occurrence Pooling \( r = 2 \) outperforms orders \( r = 1 \) and \( r = 3 \), and achieves 65.4%, 66.2%, and 66.0% MAP for signature lengths \( K^* = 180300, 320400, \) and 500500, respectively. These \( K^* \) correspond to dictionary sizes \( K = 600, 800, \) and 1000. BoW with \( r = 1 \) scores 3.8% less, that is 62.4% for \( K = 40000 \). Note that for larger visual dictionaries the coding step is computationally prohibitive, i.e. it takes 815s to code 1000 descriptors for \( K = 40000 \) on a single core of 2.3GHz AMD Opteron (1.5s for \( K = 800 \)). BoW of order \( r = 3 \) yields 65% MAP (\( K = 200 \) and \( K^* = 1353400 \)). The top score of 66.2% is attained by Second-order Occurrence Pooling with Spatial Coordinate Coding (SCC) [43]. Ignoring spatial information (i.e. no SCC) decreases MAP by 1.4%.

Figure 6(b) compares BoW \( (r = 2 \) with SCC) to FV and VLAT, as well as to BoW \( (r = 1) \) based on SPM (spatial) [30] and DoPM (dominant angle) [43] pyramids. With 66.2%, Second-order Occurrence Pooling outperforms FV by 1.9%. BoW \( (r = 1) \) with SPM or DoPM as well as VLAT attain lower scores of 62.8%, 63.6%, and 63.7%, respectively. The reported results are the top scores w.r.t. varying signature size.

The classification performance on Caltech101 is presented in figure 6(c). The settings are identical to those in figure 6(b). Second-order Occurrence Pooling scores 83.6 ± 0.4% accuracy for signature length \( K^* = 180300 \) \( (K = 600 \) atoms). This is a 2.8% improvement over FV \( (80.8 \pm 0.5\%) \) for the comparable signature length \( K^* = 163840 \). BoW \( (r = 1) \) with SPM \( (K^* = 120000) \) yields 81.5 ± 0.4% accuracy. FV and VLAT obtain their top scores of 82.2 ± 0.4% and 81.1 ± 0.7% for large signature sizes. Reducing the number of training images per class from 30 to 15 lowers the scores by around 8%. Otherwise, all trends remain consistent.

The main observations from figure 6 are that the proposed model with second-order occurrences yields the highest performance and provides an attractive trade-off between the tractability of coding and increasing signature lengths. Also, Spatial Coordinate Coding [43] attains better results than the model without spatial information for the same \( K^* \).

### 5.3 Descriptor fusion with Second-order Occurrence Pooling

This section evaluates our novel approach to descriptor fusion proposed in section 3 and illustrated in figure 3. The fusions are demonstrated for three types of descriptors, namely the grey SIFT, colour components of SIFT, and the Residual Descriptor (RD) proposed in section 3.3. The following fusion schemes are presented: a) early and b) late fusions explained in sections 3.1, c) Bi-modal Second-order Occurrence

| Dataset          | No. of classes | Training samples | Test samples | Eval. measure | State-of-the-art     | Dataset          | No. of classes | Training samples | Test samples | Eval. measure | State-of-the-art     | Dataset          | No. of classes | Training samples | Test samples | Eval. measure | State-of-the-art     | Dataset          | No. of classes | Training samples | Test samples | Eval. measure | State-of-the-art     | Dataset          | No. of classes | Training samples | Test samples | Eval. measure | State-of-the-art     | Dataset          | No. of classes | Training samples | Test samples | Eval. measure | State-of-the-art     |
|------------------|----------------|------------------|--------------|---------------|---------------------|------------------|------------------|-----------------|--------------|---------------|---------------------|------------------|------------------|------------------|-----------------|---------------|---------------------|------------------|------------------|------------------|------------------|-----------------|---------------------|------------------|------------------|------------------|-----------------|---------------|---------------------|------------------|------------------|------------------|-----------------|---------------|---------------------|------------------|------------------|------------------|-----------------|---------------|---------------------|------------------|------------------|------------------|-----------------|---------------|---------------------|------------------|------------------|------------------|-----------------|---------------|---------------------|
| PascalVOC07      | 20             | 5011             | 4952         | MAP           | [29] 66.3          | [46] 82.4        |
| Caltech101       | 102            | 15/30 (per class)| 7614/6084    | acc.           | [37] 83.4          | [46] 93.4        |
| Flower102        | 102            | 2040             | 6149         | acc.           | [48] 80.3          | [49] 91.3        |
| ImageCLEF11      | 99             | 8K (x2 flip)     | 10K          | MAP           | [42] 38.8          | [51] 91.6        |
| 15scenes         | 15             | 100 (per class)  | 2965         | acc.           | [50] 89.8          | [53] 70.1        |
| PascalVOC10AR    | 9              | 608 (x2 flip)    | 613          | MAP           | [52] 65.1          | [53] 70.1        |
| MITIndoors67     | 67             | 5360 (x2 flip)   | 1340         | acc.           | [55] 64.6          | [51] 90.3        |
| SUN397           | 397            | 19850            | 19850        | acc.           | [57] 47.2          | [51] 93.8        |

### Summary

The datasets with corresponding state-of-the-art results, as well as experimental settings for the results presented in this section.
Poolings (r = 2) outlined in section 3.2, c) Multi-modal Second-order Occurrence Pooling, e.g. fusion of all three types of descriptors. We also report results for FV and VLAT, both using the early fusion. The results in figure 7(a) include comparison of baseline approach to fusion schemes with different coders.

**Baseline** results for Second-order Occurrence Pooling with SC, LLC, and LcSA (K = 600, K = 180300) are presented without any fusion. The best MAP scores for baseline are obtained with SC (65.4%) followed by LLC (62.9%) and LcSA (58.3%). This is due to the different quantisation loss (cf. equation (49)) of the coders. We measured ξ2 according to equation (49) over a large number of descriptors, averaged the error scores, and observed the same ranking ξ2 SC < ξ2 LLC < ξ2 LcSA. We also note that the gap in performance between SC and LcSA is 7.1% for r = 2. The gap is smaller for the standard BoW (r = 1) with SPM. This shows that the quantisation noise is amplified by the higher order occurrences of visual words. The undesired effects of the quantisation loss are addressed by our fusion with Residual Descriptor.

**Residual Descriptor (RD)** is combined with SC, LLC, and LcSA by bi-modal and late fusions. Compared to the baseline, the late fusion with the RD in figure 7(a) does not make a significant difference for any of the coders. This is expected as the late fusion does not associate the residual codes with the corresponding descriptors or with the mid-level features (cf. section 3.3). The cross-term is also insufficient without the self-tensors in equations (39) and (41). However, capturing the co-occurrences of RD with the corresponding features by using bi-modal fusion results in a significant gain for all coders, i.e. 0.8%, 1.6%, and 3.3% MAP for SC, LLC, and LcSA, respectively. The benefits from using RD are larger for the coders with high quantisation loss. Note that SC attains 66.2% MAP with the overall signature length K∗ = 263536, which is reduced from K = 320400 in section 5.2 (both variants yield the same score).

**Grey and colour** components of SIFT are also fused by the proposed schemes. In figure 7(b), the proposed bi-modal fusion scores 69.2% MAP (K = 800), which improves upon grey SIFT by 3%. Note that separate dictionaries are used for the grey and colour descriptors resulting in signatures of length K∗ = 960400. The best scores for late and early fusions are 68.6% and 67.3%, respectively, both below the bi-modal fusion. Finally, FV and VLAT with the early fusion perform worse than the proposed methods and score 65.6% and 64.8%. The results of a similar experiment performed on the Flower102 set are presented in figure 7(c). The observations from Pascal VOC07 in figure 7(b) hold in this experiment except for the late fusion scoring worse than the early fusion. The results for both datasets show that the proposed bi-modal fusion offers a good trade-off between the performance and the length of signatures.

**Multi descriptor fusion.** Figure 8 presents the comparison of several fusion schemes with Second-order Occurrence Pooling of grey, colour, and Residual Descriptors on ImageCLEF11 which includes many abstract topics, e.g. party life. We therefore compare the classification performance of linear and χRBF2 kernels. Evaluations w.r.t. K∗ are studied on MITIndoors67.

In addition to the fusion approaches evaluated above, i.e. early, late, bi-modal, we also introduce multi-modal fusion that combines all grey, colour, and Residual Descriptors. Results for fusion with Fisher Vectors were obtained by early fusion as in figure 7. Results for linear kernels confirm the observations from figure 7, i.e. the bi-modal fusion with RD improves upon baseline and FV. Adding colour further improves the results in particular with the multi-modal combination of grey, colour, and Residual Descriptors. Signatures of comparable size were used for all methods. The results also show a small gain when χRBF2 is chosen over linear kernels.

Evaluations on MITIndoors67 show a marginal difference for bi-modal fusion with or without Residual Descriptor. We obtain 68.9% accuracy which is close
to 70.8% of CNN approach [51].

Further evaluations on the SUN397 dataset resulted in 44.5% and 49.0% accuracy obtained for our baseline and bi-modal Second-order Occurrence Pooling, respectively, which improves upon FV with 43.0% and 47.2% [57]. These results are below 53.5% from [51], which was obtained with a more complex CNN architecture trained on the Places and ImageNet datasets with use of multiple dataset augmentations.

5.4 Low-level Descriptor Pooling

We present the results for Second- and Third-order Occurrence Pooling \((r = 2 \text{ and } r = 3)\) on the low-level grey SIFT descriptors \((\text{raw})\), that is coder-free techniques from section 4. The raw methods are compared to our Second-order Occurrence Pooling on the mid-level features. Our third-order method uses the two stage pooling, \(i.e\). eigenvalue- and coefficient-wise Power Normalisation. The Second-order Pooling on SIFT corresponds to the approach from [31], which uses the matrix logarithm and coefficient-wise Power Normalisation \(0 < \gamma \leq 1\). For best performance, SIFT \((\text{raw})\) was combined with SCC or SPM and DoPM.

Figure 9 shows the results for PascalVOC07 and Caltech101. Our Second-order Occurrence Pooling \((r = 2)\) with the SC coder \((\text{SC+SCC})\) outperforms coder-free methods \((\text{raw+...})\). For instance, coder-free Third-order Occurrence Pooling \((r = 3)\) results in a somewhat lower performance except for Caltech101 which suggests that datasets with little clutter and well aligned objects of fixed scale can be reliably classified without the coding step and mid-level features. However, Second-order Pooling \((r = 2)\) with SPM \((\text{raw+SPM})\) from [31] yields 54.0% MAP for PascalVOC07 and 78.5% for Caltech101. These scores are nearly 10% and 6% lower than results on our coder-free approaches \((r = 3)\). Another interesting observation is the improvement introduced by our bi-modal fusion. For comparable signature lengths,
Figure 10 reveals a vital difference between the coders used in the experiments. We note that Sparse Coding with the \( \oplus n \) operator outperforms the geodesic distance, *i.e.* matrix logarithm (logm) used in [31] as well as the eigenvalue Power Normalisation combined with the MaxExp operator (ePN+MaxExp). The higher scores of SC suggest that it generates low-correlated mid-level features, which benefit from coefficient-wise pooling such as \( \oplus n \). In contrast to SC and LLC, LcSA with coefficient-wise pooling (MaxExp\( \oplus n = 40 \)) scores 3% less than logm. This significant difference can be attributed to the correlation between dimensions of mid-level features obtained with LcSA. It is demonstrated in [60] that LcSA does not take into account correlation between visual words selected for low-level descriptors whilst LLC does. Soft Assignment [7], which is closely related to LcSA [10], exhibits even larger correlations.

The above observations are supported by the simulations in figure 11. SIFT descriptors from 100 randomly selected images (PascalVOC07) were coded with the SC, LLC, LcSA, and SA coders. The same coding parameters and dictionaries were used as in experiments from figure 10. Histograms of the absolute values of Pearson’s correlation coefficients were computed. Figure 11(a) shows that for SC and LLC most of the coefficients of mid-level features are weakly correlated. In contrast, low-level descriptors (raw), LcSA, and SA produce more correlated coefficients. Figure 11(b) shows that the ePN operator reduces coefficient correlation for all methods.

### 6 CONCLUSIONS

This paper proposes a theoretically derived framework that extends Bag-of-Words with higher-order occurrences computed on mid-level features. According to our results, Second-order Occurrence Pooling offers the best trade-off between the complexity of coding, the length of signatures, and the classification performance. It outperforms the first-order variants of BoW, Fisher Vector Encoding, and Vector of Locally

### TABLE 2

Summary of the best results from this study. See figures 6-9 for detailed comparisons.

<table>
<thead>
<tr>
<th>SC</th>
<th>Uni-modal</th>
<th>Bi-modally fused SPM*</th>
</tr>
</thead>
<tbody>
<tr>
<td>---</td>
<td>-----------</td>
<td>-----------------------</td>
</tr>
<tr>
<td>r=2</td>
<td>66.2 ± 18.4</td>
<td>67.0 ± 64.4</td>
</tr>
<tr>
<td>r=3</td>
<td>62.7 ± 18.1</td>
<td>63.9 ± 8.5</td>
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<table>
<thead>
<tr>
<th>FV</th>
<th>Uni-modal</th>
<th>Bi-modally fused SPM*</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>64.0 ± 19.7</td>
<td>70.2 ± 31.7</td>
</tr>
<tr>
<td>VLAT</td>
<td>63.7 ± 8.4</td>
<td>66.1 ± 8.5</td>
</tr>
<tr>
<td>SCC, ( r=3 )</td>
<td>62.4 ± 7.8</td>
<td>67.8 ± 8.4</td>
</tr>
<tr>
<td>DDoPMf, ( r=3 )</td>
<td>62.8 ± 7.4</td>
<td>68.0 ± 8.5</td>
</tr>
<tr>
<td>DoPM</td>
<td>66.0 ± 18.5</td>
<td>69.1 ± 8.5</td>
</tr>
</tbody>
</table>

### REFERENCES


Piotr Koniusz is a post-doctoral researcher at the team LEAR, INRIA, Rhône-Alpes. He received the B.Sc. degree in Telecommunications and in Architecture and Design of Microcontroller Systems from the Warsaw University of Technology, 2004. He completed his PhD degree in Computer Vision at the University of Surrey (CVSSP) in 2013. His interests include visual concept detection, action recognition, feature and representation learning.

Fei Yan is a research fellow in the Centre for Vision, Speech and Signal Processing (CVSSP) at the University of Surrey. His research interests include machine learning and computer vision, in particular, kernel methods, distance metric learning, object recognition, and object tracking. He has publications in major machine learning and computer vision conferences and journals, including ICDM, ECML, CVPR, CVIU, PR, PAMI, and JMLR.

Philippe Henri Gosselin received the PhD degree in image and signal processing in 2005 (Cergy, France). After 2 years of post-doctoral positions at the LIP6 Lab. (Paris, France) and at the ETIS Lab. (Cergy, France), he joined the MIDI Team in the ETIS Lab as an assistant professor, and then promoted to full professor in 2012. His research focuses on machine learning for online multimedia retrieval. He developed several statistical tools for dealing with the special characteristics of content-based multimedia retrieval. This includes studies on kernel functions on histograms, bags and graphs of features, but also weakly supervised semantic learning methods. He is involved in several international research projects, with applications to image, video and 3D objects databases.

Krystian Mikolajczyk is an Associate Professor at the Centre for Vision, Speech and Signal Processing at the University of Surrey. He completed his PhD degree in 2002 at the Institute National Polytechnique de Grenoble, France. He then worked as a researcher at INRIA, University of Oxford and Technical University of Darmstadt (Germany), before joining the University of Surrey in 2005. He has more than 70 publications in top-tier computer vision and machine learning forums and received Longuet-Higgins Prize 2014 for his contribution to invariant local image descriptors. He has served in various roles at major international conferences (e.g. ICCV, CVPR, ICLR, NIPS), co-chairing British Machine Vision Conference 2012 and IEEE International Conference on Advanced Video and Signal-Based Surveillance 2013.